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On i-Separation Axioms

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Abstract—This paper is devoted to introduce a new type of separation axioms which we call i-separation axioms which depend on a new generalized open sets which called i-open sets[8], where we discuss the relation among i-separation axioms and give many examples about it. besides, We have get that separation axioms give i-separation axioms, We give examples to show that the converse may not be true. Also, On the other hand, We have proved some theorems to discuss the property of i-separation axioms.

1 INTRODUCTION

Mohammed and Askandar In 2012 [8], introduced the concept of i-open sets which they could to entire them together with many other concepts of Generalized open sets. Pervin, W., ((Foundations of General Topology)), in 1964[12], studied about ($T_\circ$, T_1, T_2, T_3, $T_{(3\frac{1}{2})}$, T_4 and T_5 spaces) (see [11] [12]) for open sets[11] by using (Klomogorov (respect. Frechet, Hausdorff, Vietoris, Urysohn and Titus Axioms)) (see [11]). In 2006 Fatima, M. Mohammad introduced Pre- Techonov and Pre-Hausdorff Separation Axioms in Intuitonistic Fuzzy special topological spaces[9] by using the concept of Pre-open sets[7] . In 2011 Y.K. Kim, R. Devi and A. Selvakumar used $\alpha\psi-Open\ sets$ [4] to introduce the concept of Weakly Ultra Separation Axioms[4]. In 2012 Al-Sheikhly, A.H. and Khudhair, H.K.[1] introduced another Type of Separation Axioms depend on an $\theta g-open\ \ set$ [3]. The aim of this paper is to introduce a new type of Separation Axioms depend on i-open stes[8] which we call i-separation axioms such as $T_{\circ i}$, T_{1i}, T_{2i}, T_{3i}, $T_{(3\frac{1}{2})i}$, T_{4i}, T_{5i} and T_{6i} spaces. This class of separation axioms may be to enter together with other separation axioms for compare and to find the similar properties and characterizations. Throughout this work (X,τ) and (X,τ^i) always are topological spaces(where τ^i is a family of all i-open sets[8] of X). This work consists of three sections. In the first one we define i-separation axioms and we give many related examples. In the second section we discuss the relation among i-separation axioms. also, the relation between T_{1i} and $Semi-T_1$ space(see Theorem2.2) and we give examples to show that the converse may not be true. Finally, in the third section we have proved some important theorems to discuss the property of i-separation axioms(see 4.4, 4.5, 4.6, 4.7, 4.10, 4.13, 4.16 and 4.17).

2 Definitions and Examples.

In this Section we define $T_{\circ i}$, T_{1i}, T_{2i}, T_{3i}, $T_{(3\frac{1}{2})i}$, T_{4i}, T_{5i} and T_{6i} spaces for i-open sets[8] and we determine them by giving many examples. Specially, We define $T_1, T_{1\alpha}$ and $Semi-T_1$ spaces to compare them with T_{1i} space.

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2.1 Definition

A subset ***A*** of a topological space (X,τ) is said to be i-open set[8] if there exists an open set $G\neq\phi$, X such that $A\subseteq CL(A\cap G)$. The complement of an i-open set is called i-closed set.

2.2 Example

Let $X=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{a, c\}, X\}$ by 2.1 we have: i-open sets are : ϕ, $\{a\}$, $\{a, c\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, X.

2.3 Definition

A mapping ***f***: $(X,\tau)\rightarrow(Y,\delta)$ is said to be i-continuous[8] at the point $x_o\in X$ if and only if for each open set $I^*\in\delta$ containing $f(x_o)$ there exists an i-open set I in (X,τ) containing $x_\circ$ such that $f(I)\subseteq I^*$. ***f*** is i-continuous map if it is i-continuous at all points of X.

2.4 Definition

A subset ***A*** of a topological space (X,τ) is said to be an α–open set[10] if $A\subseteq Int(CL(Int(A)))$. The complement of an α–open set is called α–closed set **.**

2.5 Definition

A subset A of a topological space (X,τ) is said to be a semi-open set[5] if there exists an open set **G** such that $G\subseteq A\subseteq CL(G)$ where $CL(G)$ denotes the closure of a set G [11] operator in X. The complement of a semi open set is called a semi-closed set.

2.6 Lemma [8]

Every semi-open set is i-open.

2.7 Definition

A topological space (X,τ) is said to be $T_{\circ i}$ space, if given any pair of distinct points x, y of ***X***, there exists an i-open set ***I***, containing one of them but not the other (i-Klomogorov axiom).

2.8 Example

Let $X=\{a, b\}$, $\tau=\{\phi, \{a\}, X\}$, $\tau^i=\{\phi, \{a\}, X\}$

(X,τ) and (X,τ^i) are topological spaces.

$a,b \in X\ (a \neq b)\quad \exists\{a\} \in \tau^i$ s.t. $a \in \{a\}, b \notin \{a\}$.

Therefore; (X,τ) is $T_{\circ i}$.

2.9 Definition

A topological space (X,τ) is said to be T_1 space [11] (respect. $T_{1\alpha}$, $Semi-T_1$ [6], T_{1i} space) if for any two distinct points x, y of X, there are an open set (respect. α-open, semi-open and i-open set) U containing x but not y and open set(respect. α-open, semi-open and i-open set) V containing y but not x(Frechet axiom[11] (respect. α- Frechet , semi- Frechet[6] and i-Frechet axiom).

2.10 Example

Let ***X= {a, b, c},***

$\tau = \{\phi, \{a\},\{b\},\{c\},\{a, b\},\{a, c\},[b, c\},X\}$, $\tau^\alpha = \tau^s = \tau^i = \tau$

$(X,\tau),(X,\tau^\alpha),(X,\tau^s)$ and (X,τ^i) are topological spaces.

$a,b \in X(a \neq b)\ \exists\{a\},\{b\} \in \tau,\tau^\alpha,\tau^s,\tau^i$

$s.t.\ a \in \{a\}, b \notin \{a\}, b \in \{b\}, a \notin \{b\}$

$a,c \in X(a \neq c)\ \exists\{a\},\{c\} \in \tau,\tau^\alpha,\tau^s,\tau^i$

$s.t.\ a \in \{a\}, c \notin \{a\}, c \in \{c\}, a \notin \{c\}$

$b,c \in X(b \neq c)\ \exists\{b\},\{c\} \in \tau,\tau^\alpha,\tau^s,\tau^i$

$s.t.\ b \in \{b\}, c \notin \{b\}, c \in \{c\}, b \notin \{c\}$

Therefore; (X,τ) is $T_1, T_{1\alpha}, Semi-T_1$ and T_{1i} space.

2.11 Definition

A topological space (X,τ) is said to be T_{2i} space if for any two distinct points x, y of X, there exists two separated i-open sets I_1 and I_2 such that I_1 containing x and I_2 containing y (i-Hausdorff axiom).

2.12 Definition

A topological space (X,τ) is said to be an i-Regular space(shortly R_i space) if it satisfies Vietoris axiom:[R_i] if F is an i-closed set in X and $x \in X, x \notin F$ $\exists I_1, I_2 \in \tau^i, I_1 \cap I_2 = \phi$ s.t. $F \subseteq I_1,\ x \in I_2$.

2.13 Definition

A T_{1i} space is said to be T_{3i} if its i-Regular.

2.14 Definition

A topological space (X,τ) is said to be an i-Normal space(shortly $\boldsymbol{N_i}$ space) if it satisfies i-Urysohn axiom:[$\boldsymbol{N_i}$]

$if\ F_1 \subseteq X,\ F_2 \subseteq X, F_1 \cap F_2 = \phi\ \exists I_1, I_2 \subseteq X\ s.t\ F_1 \subseteq I_1,\ F_2 \subseteq I_2$

$where\ I_1 \cap I_2 = \phi,$

$F_1, F_2\ are\ i-closed\ sets,\ I_1, I_2\ are\ i-open\ sets.$

2.15 Definition

A T_{1i} space is said to be T_{4i} if its i-Normal.

2.16 Definition

A topological space (X,τ) is said to be an i-completely regular space(shortly CR_i space) if it satisfies axiom: [CR_i] if F is an i-closed set in X and $x \in X, x \notin F$ there exist i-continuous mapping[8] $f : X \to [0,1]$ s.t. $f(F) = 1,\ f(x) = 0$.

2.17 Definition

A T_{1i} space is said to be $T_{(3\frac{1}{2})i}$ if its i-completely regular space.

2.18 Definition

A topological space (X,τ) is said to be an i-completely Normal space(shortly CN_i space) if it satisfies i-Titus axiom:[CN_i]

$if\ A_1 \subseteq X,\ A_2 \subseteq X, A_1 \cap A_2 = \phi\ \exists I_1, I_2 \subseteq X\ s.t\ A_1 \subseteq I_1,\ A_2 \subseteq I_2$

$where\ I_1 \cap I_2 = \phi, I_1, I_2\ are\ i-open\ sets.$

2.19 Definition

A T_{1i} space is said to be T_{5i} space if its i-completely Normal space.

2.20 Definition

A topological space (X,τ) is said to be an i- Perfectly Normal space(shortly $\boldsymbol{PN_i}$ space) if it satisfies the following axiom: If C_1 and C_2 are disjoint i-closed sets, there exists an i-continuous mapping $f : X \to [0,1]$ such that $f^{-1}(\{0\}) = C_1$ and $f^{-1}(\{1\}) = C_2$.

2.21 Definition

A T_{1i} space is said to be T_{6i} if its i- perfectly normal space.

2.22 Example

Let ***X= {a, b}***, $\tau = \{\phi, \{a\},\{b\},X\}$, $\tau^i = \tau$

(X,τ) and (X,τ^i) are topological spaces.

i-open sets are: ϕ, **{a},{b}**,X and i-closed sets are: ϕ, **{a},{b},*X***

$1.\ a,b \in X(a \neq b)\ \exists\{a\},\{b\} \in \tau^i$

$s.t.\ a \in \{a\}, b \in \{b\}.$

Therefore; (X,τ) is T_{1i}.

$2.\ a,b \in X(a \neq b)\ \exists\{a\},\{b\} \in \tau^i$

$s.t.\ a \in \{a\}, b \in \{b\}, \{a\} \cap \{b\} = \phi$

Therefore; (X,τ) is T_{2i}.

$3.\ \{b\}\ is\ an\ i-closed\ set\ and\ a \notin \{b\}\ there\ is\ two\ i-open sets\ \{a\},\{b\}$

$s.t.\ \ a \in \{a\}, \{b\} \subseteq \{b\}.$

Therefore; (X,τ) is i-Regular space.

4. By (1) and (3) we have: (X,τ) is T_{3i}.

$5.\ \{a\},\{b\}\ are\ i-closed\ sets\ there\ are\ two\ i-open sets\ \{a\},\{b\}$

$s.t.\ \ \{a\} \subseteq \{a\}, \{b\} \subseteq \{b\}, \{a\} \cap \{b\} = \phi.$

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Therefore; (X,τ) is i-Normal space.

6. By (1) and (5) we have: (X,τ) is T_{4i}.

7. *Let* $f : X \to [0,1]$ *be a continuous mapping and* $\{b\}$ *is an* $i-$*closed set and* $a \notin \{b\}$ *s.t.* $f(a)=0, f(\{b\})=1.$

Therefore; (X,τ) is i-Completely Regular space.

8. By (1) and (7) we have: (X,τ) is $T_{(3\frac{1}{2})i}$.

9. $\{a\},\{b\} \subseteq X$, *there are two* $i-$*opensets* $\{a\},\{b\}$ *s.t.* $\{a\} \subseteq \{a\},\{b\} \subseteq \{b\}$ *where* $\{a\} \cap \{b\} = \phi.$

Therefore; (X,τ) is i-Completely Normal space.

10. By (1) and (9) we have: (X,τ) is T_{5i}.

11. *Let* $f : X \to [0,1]$ *be an* $i-$*continuous mapping and* $\{a\},\{b\}$ *are disjoint* $i-$*closed sets s.t.* $f^{-1}(\{0\})=\{a\}, f^{-1}(\{1\})=\{b\}.$

Therefore; (X,τ) is an i- perfectly normal space

12. By (1) and (11) we have: (X,τ) is T_{6i}.

3 The Relationships among i-separation axioms

Throughout this section, we compare between $T_{\circ i}$, T_{1i}, T_{2i}, T_{3i}, $T_{(3\frac{1}{2})i}$, T_{4i} and T_{5i} spaces. Also we compare between T_{1i} space and T_1, $T_{1\alpha}$, T_{1s} spaces.

3.1 Theorem

Every T_{1i} space is $T_{\circ i}$.

3.2 Theorem

Every $Semi-T_1$ space is T_{1i} space.

The converse of 3.2 is not true.

3.3 Example

Let $X=\{1,2,3,4\}$, $\boldsymbol{\tau}=\{\phi, \{1\},\{1,2\},\{1,2,3\},X\}$,

$\tau^{\alpha} = \tau^{s} = \{\phi,\{1\},\{1,2\},\{1,2,3\},\{1,3\},\{1,4\},\{1,3,4\},\{1,2,4\},X\}$,

$\tau^{i} = \{\phi,\{1\},\{1,2\},\{1,2,3\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\}$,

$(X,\tau),(X,\tau^{\alpha}),(X,\tau^{s})$ and (X,τ^{i}) are topological spaces.

Take $1 \neq 2$:

1. There is no exists two open sets G_1,G_2 s.t. $1 \in G_1, 2 \notin G_1 2 \in G_2, 1 \notin G_2$. that is impossible. Therefore; (X,τ) is not T_1 space.

2. There is no exists two α-open sets α_1,α_2 s.t. $1 \in \alpha_1, 2 \notin \alpha_1, 2 \in \alpha_2, 1 \notin \alpha_2$. that is impossible. Therefore; (X,τ) is not $T_{1\alpha}$ space.

3. There is no exists two semi-open sets S_1,S_2 s.t. $1 \in S_1, 2 \notin S_1 2 \in S_2, 1 \notin S_2$. that is impossible. Therefore; (X,τ) is not $Semi-T_1$ space.

4. $\forall x,y \in X \ (x \neq y) \quad \exists I_1,I_2 \in \tau^{i}$ s.t. $x \in I_1, y \notin I_1 y \in I_2, x \notin I_2$

Therefore; (X,τ) is T_{1i} space.

3.4 Theorem

Every T_{2i} space is T_{1i} and also is $T_{\circ i}$.

$$T_{2i} \longrightarrow T_{1i} \longrightarrow T_{\circ i}$$

3.5 Theorem

Every CR_i space is R_i.

3.6 Theorem

Every $T_{3\frac{1}{2}i}$ space is T_{3i}.

3.7 Assistant case of i-Urysohn axiom.

A topological space (X,τ) is said to be an i-Normal space if it satisfies the following condition: For every two separated i-closed sets F_1 and F_2 in X, and for every interval $[a,b]$ of real numbers there exists an i-continuous mapping $f : X \to [a,b]$ such that: $f(F_1)=\{a\}$, $f(F_2)=\{b\}$

3.8 Theorem

Every T_{4i} space is $T_{(3\frac{1}{2})i}$.

Proof: A topological space X satisfies 2.15 of T_{4i} space, which leads to 2.17 of $T_{3\frac{1}{2}i}$ space and by 3.7, we get that the proof is complete. ■

3.9 Theorem

Every T_{5i} space is T_{4i}.

Proof: A topological space X satisfies 2.19 of T_{5i} space, which leads to 2.15 of T_{4i} space and since every two discrete i-closed sets are separated, we get that the proof is complete.■

3.10 Remark [8]

Every continuous mapping is i-continuous.

The converse of 3.10 is not true. Indeed,

Let $X=\{1,2,3,4\}$, $\tau=\{\phi,\{2\}, \{3, 4\},\{2,3,4\},X\}$, $Y=\{5,6\}$, $\delta=\{\phi,\{5\},\{6\},Y\}$,

$f: (X,\tau) \longrightarrow (Y,\delta)$, $f(1)=f(3)=f(4)=5, f(2)=6$

$\tau^{i}=\{\phi,\{2\},\{3, 4\},\{2,3,4\},\{1,3,4\},\ldots\ldots,X\}$

f is not continuous mapping but it is i-continuous mapping.

3.11 Theorem

Every subspace of T_{2i} space is T_{2i}.

Proof: Suppose X is a T_{2i} space, and A is a subspace of X. suppose that x and y are two distinct points in A and we will show that there are disjoint sets containing x and y respectively that are i-open in the subspace topology for A.

Since x and y are distinct points of X, there exists two disjoint i-open sets of X, namely U and V, such that U contains x and V contains y. suppose that $U \cap A$ and $V \cap A$ as subsets of A.

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Clearly:

• $x \in A$ and $x \in U$, so $x \in U \cap A$. Similarly, $y \in V \cap A$.

• U and V themselves are disjoint, so $U \cap A$ and $V \cap A$ are disjoint.

• $U \cap A$ is i-open relative to A, because it is the intersection with A of an i-open set in X. Similarly $V \cap A$ is also i-open relative to A. Hence, we have two disjoint i-open sets containing x and y, relative to the subspace topology of A. Therefore A is a T_{2i} space.■

3.12 Theorem

Every subspace of an i-regular space is i-regular.

Proof. Let X be an i-regular topological space and A a subset, $x \in A$ and C closed in A. now x is a point in X, D is an i-closed subset of X such that $D \cap A = C$.

Such a D exists by the way the subspace topology is defined. Clearly, whatever D is picked up for the purpose, x cannot lie in D because the only points in $D \cap A$ are in a set not containing x.

Since X is an i-regular, we can find an i-open sets U and V in X such that $x \in U$, $C \subseteq V$ and U and V are disjoint. Now, $U \cap A$ and $V \cap A$ are disjoint i-open subsets of A, with $x \in U \cap A$ and $C \subseteq V \cap A$. ■

3.13 Theorem

Every i-closed subspace of an i-normal space is i-normal.

Proof. The idea is that because the subspace is already i-closed, i-closed subsets of it are already i-closed in the whole space. So we do not have to expand the i-closed subsets.

We now separate the i-closed subsets in the whole space. We get disjoint i-open sets of the whole space. Now, simply intersect these i-open sets with the subspace, to get disjoint i-open sets of the subspace separating the two disjoint i-closed sets. ■

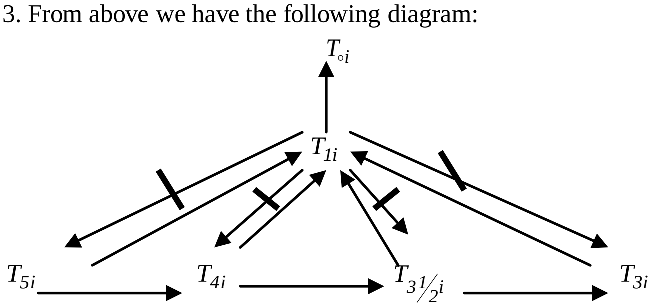
3.14 Corollaries

1. Every T_{3i} (respect. $T_{(3\frac{1}{2})i}$, T_{4i}, T_{5i} and T_{6i} space) is T_{1i} but the converse is not true because T_{1i} space is not necessary to be R_i space (respect. CR_i, N_i , CN_i and PN_i space).

2. Every $T_\circ$ space(respect. T_1, T_2, regular, T_3, completely regular, $T_{3\frac{1}{2}}$, normal, T_4, completely normal and T_5 space(see[11])) is $T_{\circ i}$ space(respect. T_{1i}, T_{2i}, i-regular, T_{3i}, i-completely regular, $T_{3\frac{1}{2}i}$, i-normal, T_{4i}, i-completely normal and T_{5i} space) but the converse is not true.

2. Proof: by 2.6 and 3.10, the proof is obtained.

3. From above we have the following diagram:



4 Many Theorems about i-separation axioms

Throughout this section, We study the i-separation axioms mentioned above by give many important theorems about them.

4.1 Definition [8]

Let (X, τ^i) be a topological space and let A be a subset of X. the intersection of all i-closed sets containing A is called i-closure of A, denoted by $CL_i(A)$: $CL_i(A) = \bigcap_{i \in \Lambda} F_i$. $A \subseteq F_i$ $\forall i$ where F_i is an i-closed set $\forall i$ in a topological space (X, τ^i). $CL_i(A)$ is the smallest i-closed set containing A.

4.2 Theorem [8]

Let (X, τ^i) be a topological space, for subsets A,B of space X, the following statements holds.

i. $CL_i(X)=X$, $CL_i(\phi) = \phi$.
ii. $CL_i(A)$ *is an i-closed set.*
iii. $A \subseteq CL_i(A)$.
iv. $A=CL_i(A)$ if and only if A is i-closed set.
v. $CL_i(A)$ is the smallest i-closed set containing A.
vi. $CL_i(A) = CL_i(CL_i(A))$.
vii. $CL_i(A \cup B) = CL_i(A) \cup CL_i(B)$.
viii. $CL_i(A) = A \cup D_i(A)$.

4.3 Theorem

A topological space (X, τ) is $T_{\circ i}$ if and only if every two different points of X have a different i-closure[8]:

$\forall x, y \in X \ (x \neq y),\ CL_i\{x\} \neq CL_i\{y\}$.

Proof: 1. Let $x \neq y$ to need $CL_i\{x\} \neq CL_i\{y\}$ for every two different points x and y in X. Since the two sets $CL_i\{x\}, CL_i\{y\}$ are different, there exist a point z in X belongs only to one of these two sets and let $z \in CL_i\{x\}, z \notin CL_i\{y\}$.

If $x \in CL_i(\{y\})$ *then* $CL_i(\{x\}) \subseteq CL_i(CL_i(\{y\})) = CL_i(\{y\})[4]$.

We have $z \in CL_i(\{x\}) \subseteq CL_i(\{y\})$ *that is contraduction*.

Then $x \notin CL_i(\{y\})$, therefore; $(CL_i(\{y\}))^C$ is an i-open set(4.2) [8] containing x not y.

2. On the other hand let X be $T_{\circ i}$ space and let x, y be two different points in X. By definition of $T_{\circ i}$ space there exists an i-open set I containing one of these two points not the other.

Let $x \in I, y \notin I$ then I^C is an i-closed set(4.2) [8] containing x not y. By definition of $CL_i(\{y\})$[8] we have $y \in Cl_i(\{y\})$ but $x \notin CL_i(\{y\})$ because $x \notin I^C$. Therefore; $CL_i\{x\} \neq CL_i\{y\}$. ■

4.4 Theorem

A topological space (X, τ) is T_{1i} if and only if every single set belongs to it is i-closed.

Proof: 1. Suppose that every single set belongs to a topological space (X, τ^i) is i-closed and x, y be two different points in X.

Then $\{x\}^C$ is an i-open set containing y not x, $\{y\}^C$ is an i-open set containing x not y. Therefore; (X, τ) is T_{1i} .

2. On the other hand, suppose that (X,τ) is T_{1i} space and $x \in X$. By definition of T_{1i} we have: for every two different points belongs to $X(x,y \in X,\ x \neq y)$ there exists an i-open set I_y containing y not x such that $y \in I_y \subseteq \{x\}^C$. Then $\{x\}^C = \bigcup\{y : y \neq x\} \subseteq \bigcup\{G_y : y \neq x\} \subseteq \{x\}^C$

Therefore; $\{x\}^C$ is the union of i-open sets, then its i-open set.
Then $\{x\}$ is i-closed set $\forall x$ in X. ■

4.5 Theorem

A topological space (X,τ) is R_i if and only if $\forall x \in X$ and for every i-open set $\boldsymbol{I}$ containing x there exists an i-open set I^* such that $x \in I^*$ *and* $CL_i(I^*) \subseteq I$.

Proof:1. suppose that (X,τ) be $\boldsymbol{R_i}$ ***space*** and let $x \in I$, *where* I *is an* $i-open$ *set in* X.
Then $F=X\backslash I$ is i-closed set not contains x.
By definition of R_i space there exists two discrete i-open sets I_x *and* I_F such that $x \in I_x$ *and* $F \subseteq I_F$.
Since $I_x \subseteq (I_F)^C$ then $CL_i(I_x) \subseteq CL_i(I_F^C) = I_F^C \subseteq F^C = I$.
Therefore $x \in I_x$ and $CL_i(I_x) \subseteq I$.
Then $\boldsymbol{I}$ is the i-open set which we want it.
2. On the other hand let the condition above is true and we will prove that X is R_i.
Let $x \notin F$ where $\boldsymbol{F}$ is an i-closed set, then $x \in F^C$ where F^C is i-open set. Then there exists an i-open set $\boldsymbol{I^*}$ such that $x \in I^*$ *and* $CL_i(I^*) \subseteq F^C$.
It is clear that I^* *and* $(CL_i(I^*))^C$ are discrete i-open sets such that $x \in I^*, F \subseteq (CL_i(I^*))^C$. Therefore; (X,τ) is R_i space ■

4.6 Theorem

A topological space (X,τ) is N_i if and only if for every i-closed set F and for every i-open set I containing F there exists an i-open set I^* such that $F \subseteq I^*$ *and* $CL_i(I^*) \subseteq I$.

Proof: 1. Suppose that (X,τ) be N_i *space* and let F be an i-closed set contained in the i-open set I then $K=X\backslash I$ *is* i-closed set(where K *and* F are discrete sets).
By definition of N_i space there exists two i-open sets I_K *and* I_F such that $K \subseteq I_K$ *and* $F \subseteq I_F$.
Since $I_F \subseteq X\backslash I_K$ then $CL_i(I_F) \subseteq CL_i(X\backslash I_K) = X\backslash I_K \subseteq X\backslash K = I$.
Therefore $x \in I_x$ and $CL_i(I_x) \subseteq I$.
Then I_F is the i-open set which we want.
2. On the other hand let the condition above is true and we will prove that X is N_i.
Let F_1 *and* F_2 be two discrete i-closed sets in X, then $F_1 \subseteq X\backslash F_2$ where $X\backslash F_2$ is i-open set.
Then there exists an i-open set I^* such that $F_1 \subseteq I^*$ and $CL_i(I^*) \subseteq X\backslash F_2$
It is clear that I^* and $X\backslash CL_i(I^*)$ are discrete i-open sets such that $F_1 \subseteq I^*, F_2 \subseteq X\backslash CL_i(I^*)$.Therefore; (X,τ) is $\boldsymbol{N_i}$ space ■

4.7 i-Urysohn's Lemma

A topological space X is an i-normal if and only if for any disjoint i-closed sets C_1 and C_2, there exists an i-continuous mapping $f : X \to [0,1]$ such that $f(C_1)=\{0\}$ and $f(C_2)=\{1\}$.

Proof**:** Let X be an i-normal space and let U and V be two i-closed sets. Set $U_\circ$ to be U, and set U_1 to be X.

Let $U_{\frac{1}{2}}$ be a set containing $U_\circ$ whose i-closure is contained in U_1. In general, inductively define for all natural numbers n and for all natural numbers $a < 2^{n-1}$

$U_{\frac{2a+1}{2^n}}$ to be a set containing $U_{\frac{a}{2^{n-1}}}$ whose i-closure is contained within the complement of $U_{\frac{a+1}{2^{n-1}}}$. This defines U_p where p is a rational number in the interval [0,1] expressible in the form $\frac{a}{2^n}$ where a and n are whole numbers.

Now define the mapping $f : X \to [0,1]$ to be $f(p)=\inf\{x \mid p \in U_x\}$.

Consider any element x within the normal space X[12], and consider any open interval *(a,b)* around *f(x)*. There exists rational numbers c and d in that open interval expressible in the form $\frac{p}{2^n}$ where p and n are whole numbers, such that *c<f(x)<d. If c<0*, then replace it with 0, and *if d>1*, then replace it with 1. Then the intersection of the complement of the set U_C and the set U_d is an open neighborhood of *f(x)* with an image within *(a, b)*, proving that the mapping is continuous. Since every continuous mapping is i-continuous mapping(remark 2.10[8]) we have: the mapping $f : X \to [0,1]$ is i-continuous.

Conversely, suppose that for any two disjoint i-closed sets, there is an i-continuous mapping f from X to [0,1] such that *f(x)=0* when x is an element of U, and that *f(x)=1* when x is an element of V. Then since the disjoint sets [0,0.5) and (0.5,1] are i-open and under the subspace topology, the inverses $f^{-1}([0,.5))$, which contains X, and $f^{-1}((.5,1])$, which contains Y, are also i-open and disjoint. ■

4.8 Remark

Let (X^*,τ^*) be a partial topological space of a topological space (X,τ) and let $F \subseteq X^*$ then $\tau^* \subseteq \tau \subseteq \tau^i$ if and only if $X^* \in \tau$.

4.9 Theorem

A topological space (X,τ) is $\boldsymbol{CN_i}$ if and only if every partial topological space of it is $\boldsymbol{N_i}$ space.

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Proof: 1. Suppose that (X,τ) be CN_i and let (X^*,τ^*) be a partial space of *X*.

Let F_1 *and* F_2 be two discrete i-closed sets in X, then:

$$F_1 \cap CL_i(F_2) = CL_i^*(F_1) \cap CL_i(F_2) = X^* \cap CL_i(F_1) \cap CL_i(F_2) = CL_i^*(F_1) \cap CL_i^*(F_2) = F_1 \cap F_2 = \phi.$$

Then F_1 *and* F_2 are separated sets in *X*.

By definition of CN_i space there exists two i-open sets I_1 *and* I_2 such that $F_1 \subseteq I_1$ *and* $F_2 \subseteq I_2$. then $X^* \cap I_1, X^* \cap I_2$ are discrete i-open sets in *X**.

Where $F_1 \subseteq X^* \cap I_1, F_2 \subseteq X^* \cap I_2$. Therefore; (X^*,τ^*) is N_i space.

2. On the other hand suppose that every partial topological space of (X,τ) is N_i and we will prove that *X* is CN_i.

Let A_1 *and* A_2 be separated sets in *X* and let the i-open set $[CL_i(A) \cap CL_i(B)]^C = X^*$ be a partial space of *X*, This space is N_i (by suppose) and $X^* \cap CL_i(A), X^* \cap CL_i(B)$ are two discrete i-closed sets in X*. Then there exists two discrete i-open sets G_A *and* G_B in *X** such that $X^* \cap CL_i(A) \subseteq G_A$, $X^* \cap CL_i(B) \subseteq G_B$.

Since *X** is an i-open set in X, then G_A *and* G_B are i-open sets in *X* too(4.8).

Then $A \subseteq X^* \cap CL_i(A) \subseteq G_A$, $B \subseteq X^* \cap CL_i(B) \subseteq G_B$.

Therefore; (X,τ) is CN_i space. ■

4.10 Theorem

Let a topological space (X,τ) be N_i space, then (X,τ) is CR_i space if and only if (X,τ) is R_i.

Proof: It is enough to prove that every N_i and R_i space is CR_i (3.5). Let $x \notin F$ where *F* is an i-closed set in *X*, then $x \in F^C$ where F^C is i-open set. Then there exists an i-open set *I** such that $x \in I^*$ *and* $CL_i(I^*) \subseteq F^C$ (4.5).

Since F *and* $CL_i(I^*)$ are discrete i-closed sets in N_i space (X,τ) and by(3.7), there exists an i-continuous mapping $f: X \to [0,1]$ such that:

$f(F) = \{1\}$, $f(CL_i(I^*)) = \{0\}$ and since $x \in I^*$ then $f(x) = 0$.

Therefore; (X,τ) is CR_i space.

4.11 Definition [8]

A mapping $f: (X,\tau) \to (Y,\delta)$ is called i-irresolute if and only if $f^{-1}(A)$ is i -closed set in (X,τ) for every i-closed set A^* in (Y, δ).

4.12 Remark

The 4.11 definition is also true on i-open sets by taking the complement.

4.13 Theorem

Let (X,τ) be a topological space and (Y,δ) is an T_{2i} space. If $f: (X,\tau) \to (Y,\delta)$ is injective[12] and i-irresolute[8], then X is T_{2i} space.

Proof: Suppose that $x, y \in X$ such that $x \neq y$.

Since f is injective, then $f(x) \neq f(y)$.

Since (Y,δ) is T_{2i} space, then there are two i-open sets I_1 and I_2 in Y such that $f(x) \in I_1$, $f(y) \in I_2$ and $I_1 \cap I_2 = \phi$.

Since f is i-irresolute then $f^{-1}(I_1), f^{-1}(I_2)$ are two i-open sets in X, $x \in f^{-1}(I_1)$, $y \in f^{-1}(I_2)$, $f^{-1}(I_1) \cap f^{-1}(I_2) = \phi$. Hence X is T_{2i} space. ■

4.14 Theorem

Let (X,τ) be a topological space and (Y,δ) is an T_{2i} space. If ***f**:* $(X,\tau) \to (Y,\delta)$ is injective[12] and i-continuous mapping[8], then *X* is T_{2i} space.

Proof: By the same way in 4.13, and using ***f*** as i-continuous mapping instead of i-irresolute mapping. ■

4.15 Definition [8]

A mapping ***f**:* $(X,\tau) \to (Y,\delta)$ is called i-closed if the image ***f***(*F*) of each closed set ***F*** in (X,τ) is i-closed set in **(***Y, δ).*

4.16 Theorem

Let (X,τ) and (Y,δ) be topological spaces and Y is an i-regular space. If ***f**:* $(X,\tau) \to (Y,\delta)$ is an i-closed mapping[8], i-irresolute and one to one, then X is an i-regular space.

Proof: Let ***F*** be a closed set in X, $x \notin F$. Since f is i-closed mapping, then $f(F)$ is i-closed set in Y. $f(x) = y \notin f(F)$. But **Y** is i-regular space, then there are two i-open sets I_1 and I_2 in Y such that $f(F) \subseteq I_2$, $y \in I_1$ and $I_1 \cap I_2 = \phi$.

Since f is i-irresolute mapping and one to one, so $f^{-1}(I_1), f^{-1}(I_2)$ are two i-open sets in *X* and $x \in f^{-1}(I_1), F \subseteq f^{-1}(I_2)$, $f^{-1}(I_1) \cap f^{-1}(I_2) = \phi$. Hence *X* is i-regular space. ■

4.17 Theorem

Let ***f*** be i-closed and i-irresolute mapping from a topological space (X,τ) into a topological space (Y,δ). if *Y* is an i-normal space, so is *X*.

Proof: Let F_1, F_2 be closed sets in *X* such that $F_1 \cap F_2 = \phi$.

Since f is i-closed mapping, then $f(F_1), f(F_2)$ are two i-closed sets in Y and $f(F_1) \cap f(F_2) = \phi$. $f(x) = y \notin f(F)$. Since Y is i-normal space and f is i-irresolute mapping, then there are two i-open sets I_1 and I_2 in Y such that $f(F_1) \subseteq I_1$, $f(F_2) \in I_2$ and $I_1 \cap I_2 = \phi$, also $f^{-1}(I_1), f^{-1}(I_2)$ are two i-open sets in *X* and $F_1 \subseteq f^{-1}(I_1), F_2 \subseteq f^{-1}(I_2)$, $f^{-1}(I_1) \cap f^{-1}(I_2) = \phi$. Hence *X* is i-normal space. ■

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